

Analytic solutions of initial-boundary-value problems of transient conduction using symmetries

H. Azad¹, M. T. Mustafa¹ and A. F. M. Arif²

¹Department of Mathematics and Statistics,
King Fahd University of Petroleum and Minerals,
Dhahran 31261, Saudi Arabia

hassanaz@kfupm.edu.sa, tmustafa@kfupm.edu.sa

²Department of Mechanical Engineering,
King Fahd University of Petroleum and Minerals,
Dhahran 31261, Saudi Arabia

afmarif@kfupm.edu.sa

Abstract

Lie symmetry method is applied to find analytic solutions of initial-boundary-value problems of transient conduction in semi-infinite solid with constant surface temperature or constant heat flux condition. The solutions are obtained in a manner highlighting the systematic procedure of extending the symmetry method for a PDE to investigate BVPs of the PDE. A comparative analysis of numerical and closed form solutions is carried out for a physical problem of heat conduction in a semi-infinite solid bar made of AISI 304 stainless steel.

1 Introduction

Lie symmetry method is a powerful general technique for analyzing non-linear PDEs and can be efficiently employed to study problems having implicit or explicit symmetries. Thus it provides most widely applicable technique to find the closed form solutions of differential equations and contains, as particular case cf. [20], many efficient methods for solving differential equations like separation of variables, traveling wave solutions, self-similar solutions and exponential self-similar solutions. Since the modern treatment of the classical Lie symmetry theory by Ovsiannikov [19], the theory of symmetries of differential equations has been studied intensely and has substantially grown. A large amount of literature about the classical Lie symmetry theory, its applications and its extensions is available, e.g. [1, 2, 5, 6, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21].

Most of the engineering and physical problems require the PDE to be solved subject to suitable initial or boundary conditions. Although there have been some notable

contributions in the applications of symmetry method to boundary value problems (BVPs) c.f. [1, 3, 7, 8, 19], in general the Lie symmetry method has not been utilized in a great deal in obtaining solutions of BVPs of physical significance. One reason could be the natural restrictions imposed because of the requirement of invariance of initial-boundary conditions in addition to requirement of invariance of PDE under the symmetry. This work is concerned with application of Lie symmetry method to obtain analytic solution of two standard initial-boundary-value problems of heat conduction, in a systematic manner that highlights the systematic procedure of extending the symmetry method for a PDE to investigate BVPs of the PDE.

A description of the initial-boundary value problems, under study, is provided in Section 2. Section 3 is divided in two parts, giving an introduction to the solution-method followed by its application to obtain exact solutions of the initial-boundary value problems described in Section 2. The comparative analysis of numerical and analytic solutions for the heat conduction in solid bar made of AISI 304 stainless steel is presented in Section 4.

2 Description of the initial-boundary value problems

We consider test problems related to transfer of heat by conduction. The analysis of such problems is required in many physical engineering problems, for example, the cooling of electronic equipment, the design of thermal-fluid systems, and the material and manufacturing processes. In practice, a major objective of the solution of such problems is to determine the temperature field in a medium as a result of either thermal condition applied to the boundary of the medium or heat generation within the medium. Once the exact temperature distribution is known, the heat flux at any point in the system, including the boundaries, can be computed from the Fourier's law.

The problems studied here are for the transient conduction in semi-infinite solid either with constant surface temperature or with constant surface heat flux conditions. Precisely, we investigate the following initial-boundary value problems.

IBVP-1

Transient conduction in a semi-infinite solid with constant surface temperature

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad (2.1)$$

with initial and boundary conditions

$$T|_{t=0} = T_i, \quad T|_{x=0} = T_s, \quad T|_{x \rightarrow \infty} = T_i \quad (2.2)$$

IBVP-2

Transient conduction in a semi-infinite solid with constant surface heat flux

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad (2.3)$$

with initial and boundary conditions

$$T|_{t=0} = 0, \quad -k \frac{\partial T}{\partial x}|_{x=0} = q_0'', \quad T|_{x \rightarrow \infty} = 0 \quad (2.4)$$

3 Lie symmetry solution of the initial-boundary value problems

The general procedure of applying Lie symmetry method to study IBVPs of PDEs requires [4] determination of a one parameter Lie group of transformations that leaves the problem invariant, and the utilization of these transformations either to construct the invariant solution or to obtain similarity reductions. It can be explained by the following steps:

- *Lie symmetries of the PDE*

Determining the symmetry algebra of the governing PDE.

- *Invariance of the boundaries*

Taking the most general symmetry operator X obtained in step 1 and finding the conditions under which it leaves the boundaries invariant.

- *Invariance of boundary conditions on the boundary*

Finding the restrictions on X that are imposed due to invariance of boundary conditions on the boundary.

These steps will determine the symmetry operator that leaves the IBVP invariant.

- *Construction of similarity solution or reductions*

Utilizing the similarity variables of the symmetry operator of the IBVP to find the similarity reductions and similarity solution of IBVP.

Further details about these steps are illustrated in the subsections below. It should be noted that the more general the symmetry operator (leaving the IBVP invariant) found the more likely it is to lead to the solution of the problem [16].

3.1 Lie symmetries of PDE (2.1)

The method of obtaining the classical Lie symmetries of a PDE is standard which is described in detail in many books, e.g. [6, 11, 15, 18, 21]. For instance, to obtain Lie symmetries of an equation like (2.1), one considers the 1-parameter Lie group of infinitesimal transformations in (x, t, T) given by

$$x^* = x + \epsilon \xi(x, t, T) + O(\epsilon^2)$$

$$t^* = t + \epsilon \tau(x, t, T) + O(\epsilon^2)$$

$$T^* = T + \epsilon \phi(x, t, T) + O(\epsilon^2)$$

where ϵ is the group parameter, hence the corresponding generator of the Lie algebra is of the form

$$X = \xi(x, t, T) \frac{\partial}{\partial x} + \tau(x, t, T) \frac{\partial}{\partial t} + \phi(x, t, T) \frac{\partial}{\partial T}.$$

If $X^{[2]}$ denotes the second prolongation of X then using the invariance condition

$$X^{[2]}(T_t - \alpha T_{xx})|_{T_t = \alpha T_{xx}} = 0$$

yields an overdetermined system of linear PDEs in ξ , τ and ϕ called determining equations. The general solution of this system determines the generator of the symmetry algebra. For most type of problems, Computer Algebra Systems like Macsyma, Maple, Mathematica, MuPAD and Reduce can be used either to perform above steps separately or to find the symmetry algebra directly. A detailed survey of software that can be used for symmetry analysis is given by Hereman in [14, pages 367-413].

The symmetry algebra of PDE (2.1) is well known (in fact it was found by Lie) and is spanned by the vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, & X_4 &= 2t \frac{\partial}{\partial x} - \frac{1}{\alpha} x T \frac{\partial}{\partial T}, \\ X_5 &= 4t^2 \frac{\partial}{\partial t} + 4xt \frac{\partial}{\partial x} - \frac{1}{\alpha} (x^2 + 2\alpha t) T \frac{\partial}{\partial T}, & X_6 &= T \frac{\partial}{\partial T}, & X_\infty &= f(t, x) \frac{\partial}{\partial T}. \end{aligned}$$

The corresponding optimal system of 1-dimensional subalgebras is given by the following vector fields, [18].

$$X_2, \quad X_6, \quad X_1 + cX_6, \quad X_1 + X_4, \quad X_1 - X_4, \quad X_1 + X_5 + cX_6, \quad X_3 + cX_6$$

and representatives of the form X_∞ . This means any invariant solution of PDE (2.1) can be found via a suitable transformation of the invariant solutions obtained from the symmetry operators in the optimal system. Instead, for our purpose we directly find the symmetry operator that preserves the boundary and boundary conditions and hence directly leads to the solutions of initial-boundary-value problems (see details below).

3.2 Solution to IBVP-1

We consider the general symmetry operator

$$X = k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 + k_5X_5 + k_6X_6$$

of PDE (2.1) and search for the operator that preserves the boundary and the boundary conditions (2.2).

The invariance of the boundaries $x = 0, t = 0$ or equivalently

$$\begin{aligned} [X(x - 0)]_{x=0} &= 0, \\ [X(t - 0)]_{t=0} &= 0 \end{aligned}$$

implies

$$k_1 = k_2 = k_4 = 0. \tag{3.1}$$

Hence, X must be

$$X = k_3X_3 + k_5X_5 + k_6X_6.$$

In addition to the restrictions imposed by Equation (3.1), the invariance of initial and boundary conditions i.e.

$$\begin{aligned} [X(T - T_i)]_{t=0} &= 0, \quad \text{on } T = T_i \\ [X(T - T_s)]_{x=0} &= 0, \quad \text{on } T = T_s \end{aligned}$$

implies we must have

$$k_5 = 0 = k_6. \tag{3.2}$$

Hence the IBVP (2.1), (2.2) is invariant under the symmetry

$$X = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x},$$

where we have chosen $k_3 = 1$.

The invariant solution of the problem is constructed by utilizing the transformations through similarity variables for X . Solving the characteristic system for $XI = 0$ gives $I_1 = \frac{x^2}{t}$ and $I_2 = T$ as the differential invariants of $X = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$. Hence, the similarity variables for X are

$$\xi(x, t) = \frac{x^2}{t} \text{ and } V(\xi) = T. \quad (3.3)$$

Substitution of similarity variables in Equation (2.1) implies that the corresponding similarity solution of PDE (2.1) is of the form $T = V(\xi)$ where $V(\xi)$ satisfies the ODE

$$4\xi \frac{d^2V}{d\xi^2} + \left(2 + \frac{\xi}{\alpha}\right) \frac{dV}{d\xi} = 0. \quad (3.4)$$

The above equation can be integrated, using the substitution

$$W = \frac{dV}{d\xi},$$

to obtain

$$V(\xi) = c_1 \int \frac{e^{-\xi/4\alpha}}{\sqrt{\xi}} d\xi + c_2. \quad (3.5)$$

Making the change of variable

$$y^2 = \frac{\xi}{4\alpha} \quad (3.6)$$

in the above solution yields

$$\begin{aligned} V &= 4c_1\sqrt{\alpha} \int e^{-y^2} dy + c_2 \\ &= 2c_1\sqrt{\alpha}\sqrt{\pi} \cdot \mathbf{erf}(y) + c_2, \end{aligned}$$

where \mathbf{erf} denotes the error function. Hence, from Equations (3.3), (3.5) and (3.6), the exact solution of PDE (2.1) that is invariant under $X = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$ is

$$T(x, t) = 2c_1\sqrt{\alpha}\sqrt{\pi} \cdot \mathbf{erf}\left(\frac{x/\sqrt{t}}{2\sqrt{\alpha}}\right) + c_2. \quad (3.7)$$

Imposing the initial and boundary conditions determines

$$c_1 = \frac{T_i - T_s}{2\sqrt{\alpha\pi}} \text{ and } c_2 = T_s,$$

giving the solution

$$T(x, t) = (T_i - T_s) \operatorname{erf} \left(\frac{x}{2\sqrt{\alpha t}} \right) + T_s$$

of the IBVP (2.1), (2.2).

3.3 Solution to IBVP-2

Taking the general symmetry operator

$$X = k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 + k_5 X_5 + k_6 X_6$$

of PDE (2.3), we determine the operator that leaves the boundary and the boundary conditions (2.4) invariant.

The invariance of the boundaries $x = 0, t = 0$ or equivalently

$$\begin{aligned} [X(x - 0)]_{x=0} &= 0, \\ [X(t - 0)]_{t=0} &= 0 \end{aligned}$$

yields

$$k_1 = k_2 = k_4 = 0, \tag{3.8}$$

i.e. X must be

$$X = k_3 X_3 + k_5 X_5 + k_6 X_6.$$

Since

$$[X(T - 0)]_{t=0} = 0 \tag{3.9}$$

implies

$$\left(\frac{-k_5}{\alpha} x^2 + k_6 \right) T = 0, \tag{3.10}$$

the invariance of the condition $T|_{t=0} = 0$ at $T = 0$ does not impose any restrictions on X .

The invariance of the boundary condition

$$\left[-k \frac{\partial T}{\partial x} = q_0'' \right]_{x=0}$$

requires, see [4, Chapter 4] for details,

$$\left[X^{[1]} \left(k \frac{\partial T}{\partial x} + q_0'' \right) \right]_{x=0} = 0 \quad \text{on} \quad -k \frac{\partial T}{\partial x} = q_0''$$

where $X^{[1]}$ denotes the first prolongation of X . It follows that

$$\left[(-6k_5t + k_6 - k_3) \frac{\partial T}{\partial x} \right]_{x=0} = 0$$

on

$$k \frac{\partial T}{\partial x} + q_0'' = 0,$$

hence we must have

$$k_5 = 0 \text{ and } k_3 = k_6.$$

Choosing $k_3 = k_6 = 1$ provides the symmetry

$$X = X_3 + X_6 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + T \frac{\partial}{\partial T}$$

that leaves the IBVP (2.3), (2.4) invariant.

To find the similarity transformations that will lead to the solution, the characteristics system for $XI = 0$ is solved. This provides the similarity variables

$$\xi(x, t) = \frac{x^2}{t} \text{ and } V(\xi) = \frac{T}{x}. \quad (3.11)$$

Substituting similarity variables in PDE (2.3) implies that the corresponding similarity solution is of the form

$$T = xV(\xi) \quad (3.12)$$

where $V(\xi)$ satisfies the ODE

$$4 \frac{d^2 V}{d\xi^2} + \left(\frac{6}{\xi} + \frac{1}{\alpha} \right) \frac{dV}{d\xi} = 0. \quad (3.13)$$

Following a procedure similar to the solution of ODE (3.4) and using Equations (3.11), (3.12) we obtain

$$T(x, t) = -2c_1 \sqrt{t} e^{\frac{-x^2}{4\alpha t}} - c_1 x \sqrt{\frac{\pi}{\alpha}} \cdot \mathbf{erf} \left(\frac{x/\sqrt{t}}{2\sqrt{\alpha}} \right) + c_2 x. \quad (3.14)$$

Imposing the boundary conditions determines

$$c_1 = -\frac{q_0''}{\sqrt{k}} \sqrt{\frac{\alpha}{\pi}} \text{ and } c_2 = -\frac{q_0''}{k}.$$

giving the solution

$$T(x, t) = 2 \frac{q_0''}{k} \sqrt{\frac{\alpha t}{\pi}} e^{\frac{-x^2}{4\alpha t}} + \frac{q_0''}{k} x \left\{ \mathbf{erf} \left(\frac{x}{2\sqrt{\alpha t}} \right) - 1 \right\}$$

of the IBVP (2.3), (2.4).

Remarks

1. Although the above procedure illustrates that the symmetry algebras admitted by boundary value problems, in general, may not be rich enough to obtain symmetry solutions or reductions of the BVP, it is worth noting that sometimes the invariant solutions obtained from PDE can satisfy boundary conditions that are not left invariant by the operator which generated the solution. Examples can easily be constructed from above boundary value problems. For instance, we can see from above that the invariant solution

$$T(x, t) = T_i \operatorname{erf} \left(\frac{x/\sqrt{t}}{2\sqrt{\alpha}} \right)$$

obtained from the generator

$$X_3 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x},$$

satisfies the IBVP

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

with initial and boundary conditions

$$T|_{t=0} = T_i, \quad T|_{x=2\sqrt{\alpha}} = T_i \operatorname{erf} \left(\frac{1}{\sqrt{t}} \right),$$

but the boundary condition

$$\left[T = T_i \operatorname{erf} \left(\frac{1}{\sqrt{t}} \right) \right]_{x=2\sqrt{\alpha}}$$

is not preserved by the generator

$$X_3 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}.$$

2. In case of non-linear problems it is likely that reduced ODE, obtained through similarity variables, may not be integrable in terms of known functions. In such cases, the reduced BVP of ODE either can be solved numerically or a Lie plane analysis, cf. [8], of the reduced ODE has to be carried out to understand the solution.

4 Comparison of numerical and analytic results for a physical problem

To compare the exact solution of the two boundary value problems discussed in the previous sections with numerical results using finite element method, transient heat conduction in a semi-infinite solid bar was solved. The solid is made of AISI 304 stainless steel and initially at a temperature T_i . The material properties for AISI 304 steel used in this work are given below:

$$\text{Thermal conductivity } (k) = 18.2 \text{ W/m } ^0K$$

$$\text{Specific heat } (c) = 0.536 \text{ kJ/kg } ^0K$$

$$\text{Density } (\rho) = 7822 \text{ kg/m}^3$$

$$\text{Thermal diffusivity } (\alpha = k/\rho c) = 4.34E - 3 \text{ m}^2/\text{s}$$

The spatial temperature distribution at different times was calculated when the solid is subjected to the following boundary conditions:

For BVP-1	For BVP-2
Initial temperature $T_i = 300 \text{ } ^0K$	Initial temperature $T_i = 0 \text{ } ^0K$
Surface temperature $T_s = 900 \text{ } ^0K$ at $x = 0$	Heat flux $= 5 \text{ kW/m}^2$ at $x = 0$

In order to numerically solve this problem, a transient thermal analysis is performed using FEA software ANSYS. The given structure is modeled using 3-D Conduction Bar Elements (LINK33). LINK33 is a uniaxial element with the ability to conduct heat between its nodes. The element has a single degree of freedom, temperature, at each node point. The conducting bar is applicable to a steady-state or transient thermal analysis. A refined uniform mesh is used to model the nonlinear thermal gradient through the solid. The length of the model is taken as $L = 2m$ for BVP-1 and $L = 10m$ for BVP-2 assuming that no significant temperature change occurs at the interior end point during the time period of interest. This assumption is validated by the temperature of node at $x = L$ at the end of the transient analysis.

In the figures given at the end, Figure 1 shows the spatial distribution of temperature using closed form solution of the BVP-1. The comparison of this solution with the numerical results using FEA at three different times is shown in Figure 2. Similarly, the temperature distribution for the BVP-2 from the exact solution is given in Figure-3. Figure-4 indicates very good agreement between closed form solution and numerical results for the BVP-2.

5 conclusion

This paper presents a systematic approach for applying Lie symmetry method to investigate boundary value problems of partial differential equations. Equations for transient conduction in semi-infinite solid are considered subject to either constant surface temperature condition or constant heat flux condition. Those symmetries are determined that leave the partial differential equation, boundaries and the boundary conditions invariant. The corresponding similarity transformations are then utilized to transfer the problem to boundary value problem of ordinary differential equations which are solved to obtain explicit analytic solutions in both cases. Finally, the results are applied to practical problems of heat conduction in a solid bar made up of AISI 304 stainless steel. For these particular problems, a comparative analysis of numerical and closed form solutions is also carried out. The recovery of well-known solutions of important transient conduction BVPs of engineering shows that a systematic application of Lie symmetry method to BVPs is an applicable technique and demonstrates its potential for investigating BVPs related to practical applications in different fields of applied sciences and technology.

Acknowledgments

The authors acknowledge the support of King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia.

References

- [1] W. F. Ames, *Nonlinear partial differential equations in engineering. Vol. I & II* (Academic Press, New York, 1965-1972).
- [2] G. Baumann, *Symmetry analysis of differential equations with Mathematica* (Springer-Verlag, New York, 2000).
- [3] G. W. Bluman, Applications of the general similarity solution of the heat equation to boundary-value problems, *Quart. Appl. Math.* **31**, 403 (1974).
- [4] G. W. Bluman and S. C. Anco, *Symmetry and integration methods for differential equations* (Springer-Verlag, New York, 2002).

- [5] G. W. Bluman and J. D. Cole, *Similarity methods for differential equations* (Springer-Verlag, New York, 1974).
- [6] G. W. Bluman and S. Kumei, *Symmetries and differential equations* (Springer-Verlag, New York, 1989).
- [7] Brian. J. Cantwell, *Introduction to symmetry analysis* (Cambridge University Press, Cambridge, 2002).
- [8] L. Dresner, *Similarity solutions of nonlinear partial differential equations* (Research notes in Mathematics, No. 88, Pitman, Boston, MA. 1983).
- [9] N. Euler and W. H. Steeb, *Continuous symmetries, Lie algebras and differential equations* (Bibliographisches Institut, Mannheim, 1992).
- [10] A. G. Hansen, *Similarity analyses of boundary value problems in engineering* (Prentice Hall, Englewood Cliffs, 1964).
- [11] P. E. Hydon, *Symmetry methods for differential equations* (Cambridge University Press, Cambridge, 2000).
- [12] N. H. Ibragimov (Editor), *CRC handbook of Lie group analysis of differential equations, Vol. 1, Symmetries, exact solutions and conservation laws* (CRC Press, Boca Raton, 1994).
- [13] N. H. Ibragimov (Editor), *CRC handbook of Lie group analysis of differential equations, Vol. 2, Applications in engineering and physical sciences* (CRC Press, Boca Raton, 1995).
- [14] N. H. Ibragimov (Editor), *CRC handbook of Lie group analysis of differential equations, Vol. 3, New trends in theoretical developments and computational methods* (CRC Press, Boca Raton, 1996).
- [15] N. H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations* (John Wiley & Sons, Chichester, 1999).
- [16] N. H. Ibragimov, Personal communication in MOGRAN-11, Karlskrona Sweden. (May 28-June 1, 2007).
- [17] W. Miller, *Symmetry and separation of variables* (Addison Wesley, Reading, Massachusetts, 1977).

- [18] P. J. Olver, *Applications of Lie groups to differential equations* (Springer-Verlag, New York, 1986).
- [19] L. V. Ovsiannikov, *Group analysis of differential equations* (Academic Press, New York, 1982).
- [20] A. D. Polyanin and V. F. Zaitsev, *Handbook of nonlinear partial differential equations* (CRC Press, Boca Raton-London, 2004).
- [21] H. Stephani, *Differential equations. Their solution using symmetries* (Cambridge University Press, Cambridge, 1989).

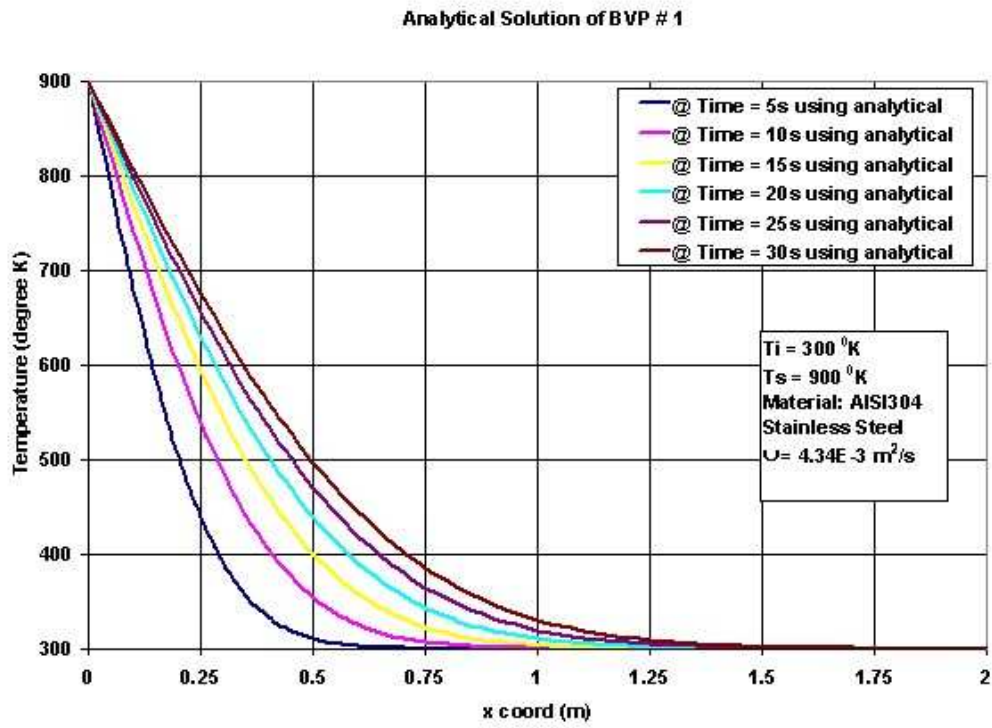


Figure 1: Temperature distribution at different times using analytical solution for BVP-1

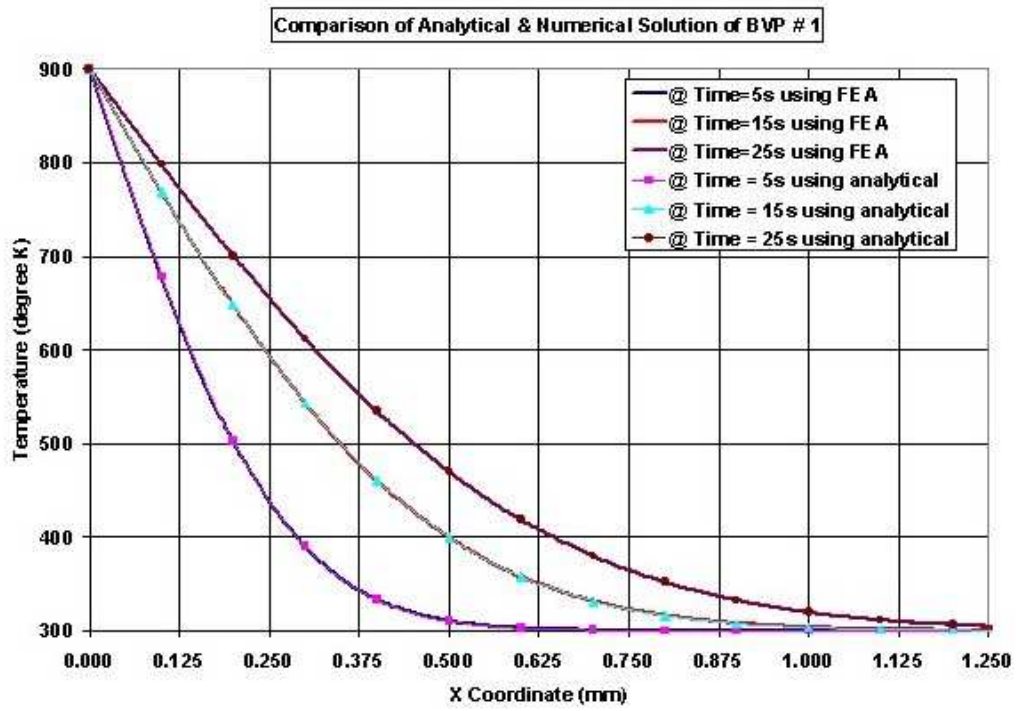


Figure 2: Comparison of analytical and numerical results for BVP-1

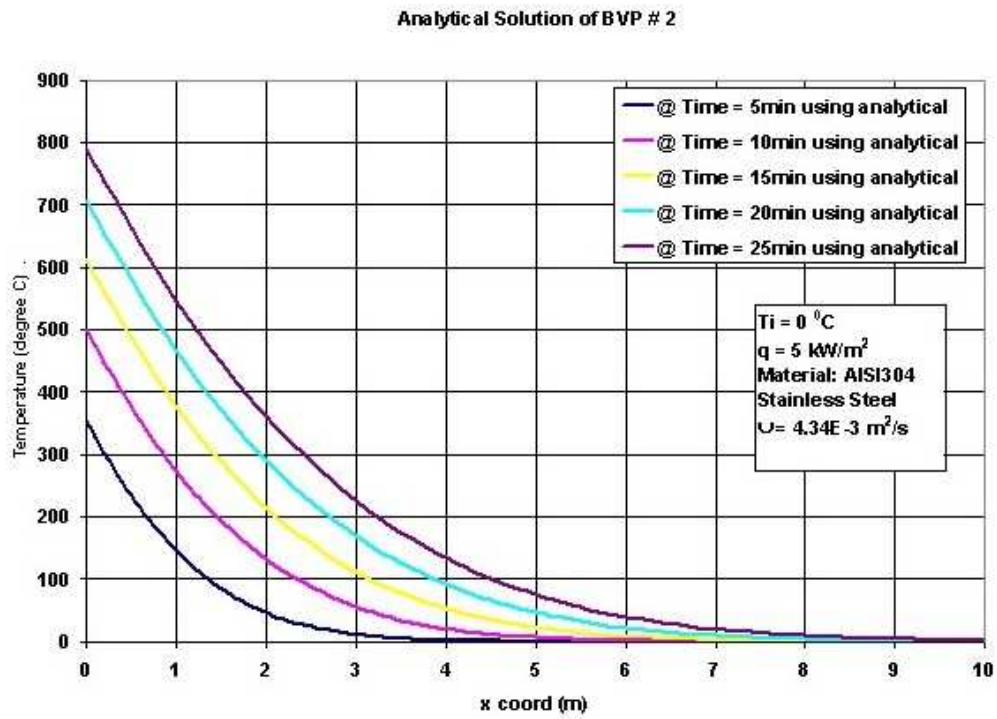


Figure 3: Temperature distribution at different times using analytical solution for BVP-2

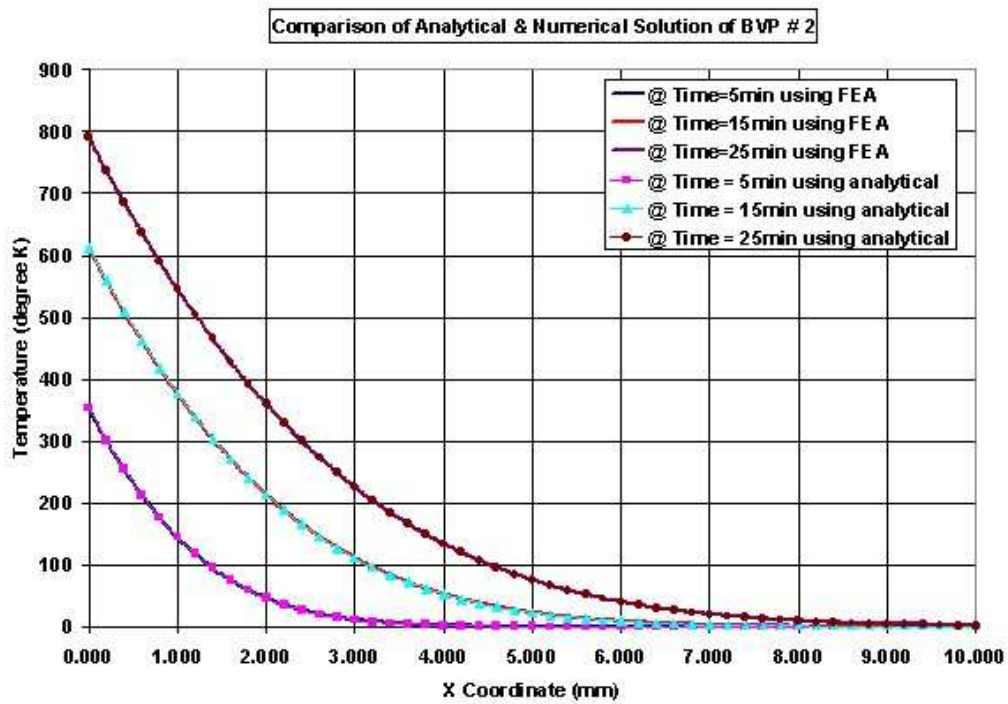


Figure 4: Comparison of analytical and numerical results for BVP-2